

The origin of the Schott term in the electromagnetic self force of a classical point charge

Michael R. Ferris and Jonathan Gratus¹

Physics Department, Lancaster University, LA1 4YB, UK

& The Cockcroft Institute, UK

The Schott term is the third order term in the electromagnetic self force of a charged point particle. The self force may be obtained by integrating the electromagnetic stress-energy-momentum tensor over the side of a narrow hypertube enclosing a section of worldline. This calculation has been repeated many times using two different hypertubes known as the Dirac Tube and the Bhabha Tube, however in previous calculations using a Bhabha Tube the Schott term does not arise as a result of this integration. In order to regain the Lorentz-Abraham-Dirac equation many authors have added an ad hoc compensatory term to the non-electromagnetic contribution to the total momentum. In this article the Schott term is obtained by direct integration of the electromagnetic stress-energy-momentum tensor.

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I. INTRODUCTION

The self force on an accelerating charged particle is the force due to the particles own electromagnetic field. The method used to derive the self force on a single particle depends on the model on which the particle is based. In 1938 Dirac¹ proposed a method based on the point particle model. In this approach the self force can be calculated directly from the electromagnetic stress-energy-momentum tensor associated with the Liénard-Wiechert field, and is found to be the sum of three terms. Two of these terms are finite, of which one is known as the ‘radiation reaction term’ and the other is known as the ‘Schott term’. The third term is infinite. It is customary to treat this infinite term as an electromagnetic contribution to mass, which combined with the ‘bare mass’ gives the observed mass of the particle. The combining of electromagnetic and bare masses is known as ‘mass renormalization’ and leads to a finite expression for the mass renormalized self force, which is the correction term in the equation of motion.

Let $C : I \subset \mathbb{R} \rightarrow \mathcal{M}$ be the proper time parameterized inextendible worldline of a point particle of mass m , bare mass m_0 and charge q where \mathcal{M} is Minkowski spacetime with metric g of signature $(-, +, +, +)$ and Levi-Civita connection ∇ where

$$\dot{C} = C_*(d/d\tau), \quad \ddot{C} = \nabla_{\dot{C}}\dot{C}, \quad \ddot{\ddot{C}} = \nabla_{\dot{C}}\nabla_{\dot{C}}\dot{C} \quad (1)$$

and $\tau \in I$. We use the SI unit convention but with the speed of light $c = 1$. It follows

$$g(\dot{C}, \dot{C}) = -1, \quad (2)$$

and hence

$$g(\dot{C}, \ddot{C}) = 0, \quad \text{and} \quad g(\dot{C}, \ddot{\ddot{C}}) = -g(\ddot{C}, \ddot{C}). \quad (3)$$

Within the point model framework the instantaneous electromagnetic 4-momentum arises as an integral of the electromagnetic stress-energy-momentum tensor over a suitable 3-surface in spacetime. In Dirac’s calculation the surface is the side Σ_T^D of a thin tube, of spatial radius R_0^D , enclosing a section of the worldline C . See FIG. 1. Since the displacement vector Y defining the Dirac tube is spacelike, the Liénard-Wiechert potential is written as a series expansion in proper time. When using a Dirac tube the integration of the electromagnetic stress-energy-momentum tensor gives for the self force¹⁻⁴

$$f_{\text{self}}^D = \kappa \left(\frac{2}{3}(\ddot{\ddot{C}} - g(\ddot{C}, \ddot{C})\dot{C}) - \lim_{R_0^D \rightarrow 0} \frac{1}{2R_0^D} \ddot{C} \right), \quad (4)$$

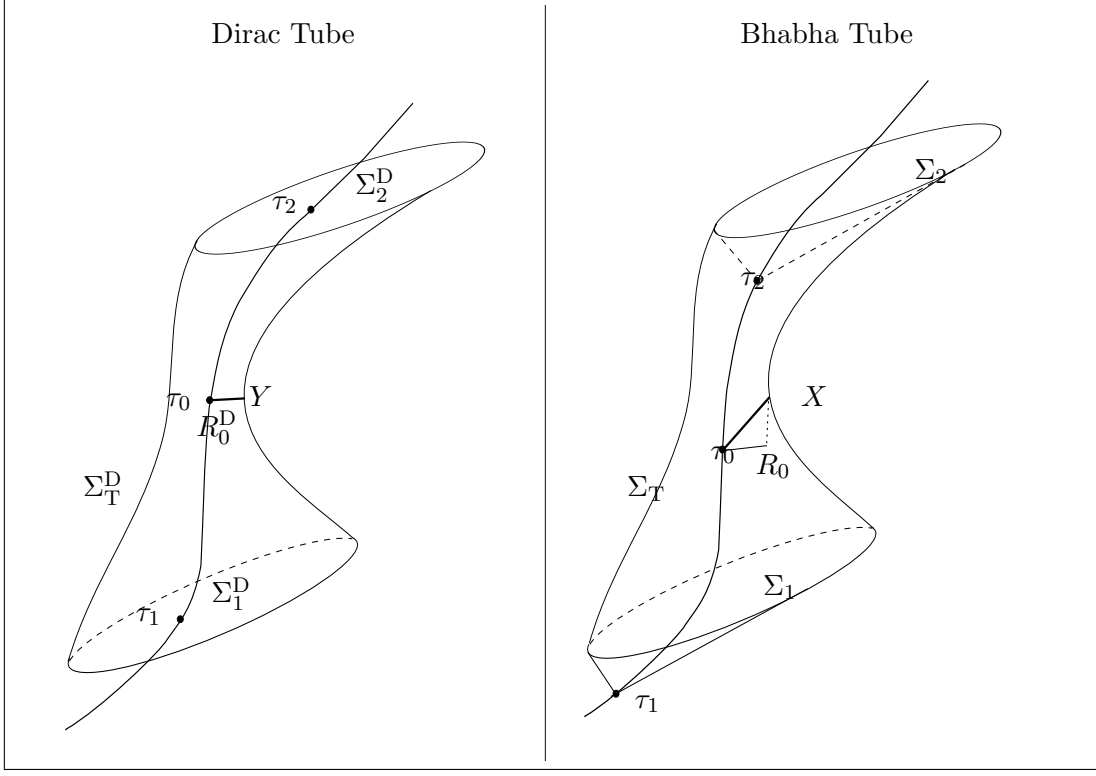


FIG. 1. The Dirac and Bhabha Tubes

where

$$\kappa = \frac{q^2}{4\pi\epsilon_0}. \quad (5)$$

The first term is known as the ‘Schott term’ and the second term is sometimes called the ‘radiation reaction’ term. The third term is the singular term whose coefficient will later be identified as an electromagnetic mass.

An alternative approach, proposed by Bhabha⁵ in 1939, is to integrate the electromagnetic stress-energy-momentum tensor over the side Σ_T of the Bhabha tube with spatial radius R_0 . The principal advantage of this approach is that the displacement vector X is lightlike and as a result the Liénard-Wiechert potential, and the corresponding electromagnetic field and stress-energy-momentum tensor, can be written explicitly. However previous articles which use a Bhabha tube to evaluate the self force give the following expression⁴⁻⁸

$$f_{\text{self}}^B = -\kappa \left(\frac{2}{3}g(\ddot{C}, \ddot{C})\dot{C} + \lim_{R_0 \rightarrow 0} \frac{1}{2R_0}\ddot{C} \right) = f_{\text{self}}^D - \frac{2}{3}\kappa\ddot{C}. \quad (6)$$

Thus the Schott term is missing, indicating a major drawback of these approaches.

In 2006 Gal'tsov and Spirin³ draw attention to this discrepancy. They claim the Schott term should arise directly from the electromagnetic stress-energy-momentum tensor and provide a derivation using Dirac's space-like coordinate system in order to show this. However they propose the missing term in (6) is a consequence of the light-like coordinates used to define the Bhabha tube. We show the term may be obtained using light-like coordinates and therefore that the missing term results from the procedure followed and not from the nature of the coordinates.

Calculation of the self force requires a minimum of two limits to be taken; the shrinking of the hypertube onto the worldline and the bringing together of the caps. In previous calculations⁴⁻⁸ the Schott term remains unnoticed because the former limit is taken before the latter. In this article the calculation of the self force requires three limits to be taken, the shrinking of the Bhabha tube Σ_T onto the worldline C i.e. $R_0 \rightarrow 0$, and the bringing together of the lightlike caps Σ_1 and Σ_2 onto the lightlike cone with vertex $C(\tau_0)$ i.e. $\tau_1 \rightarrow \tau_0$ $\tau_2 \rightarrow \tau_0$, where τ_0 is the proper time of the point where we wish to evaluate the self force (see FIG.1). We therefore have the freedom to choose the order of these limits. We choose to let the three limits take place simultaneously, subject to the constraint that

$$\lambda = \lim_{\substack{R_0 \rightarrow 0 \\ \tau_1 \rightarrow \tau_0 \\ \tau_2 \rightarrow \tau_0}} \left(\frac{\tau_1 + \tau_2 - 2\tau_0}{4R_0} \right) \quad (7)$$

where $\lambda \in \mathbb{R}$ is finite. This gives the self force as

$$f_{\text{self}} = -\kappa \left(\frac{2}{3} g(\ddot{C}, \ddot{C}) \dot{C} + \lambda \ddot{C} + \lim_{R_0 \rightarrow 0} \frac{1}{2R_0} \ddot{C} \right) \quad (8)$$

which is in agreement with f_{self}^D if $\lambda = -\frac{2}{3}$, hence the Schott term arises by direct integration of the electromagnetic stress-energy-momentum tensor.

We suppose a balance of momentum

$$\dot{P}_{\text{PART}} + \dot{P}_{\text{EM}} = f_{\text{ext}} \quad (9)$$

where total momentum has been separated into electromagnetic contribution P_{EM} and non-electromagnetic contribution P_{PART} , and $\dot{P} = \nabla_{\dot{C}} P$. All the external forces acting on the particle are denoted by f_{ext} , and in the following we show $\dot{P}_{\text{EM}} = -f_{\text{self}}$. A suitable choice for the non-electromagnetic momentum P_{PART} has to be made. Most external forces f_{ext} , including the Lorentz force, are orthogonal to \dot{C} :

$$g(f_{\text{ext}}, \dot{C}) = 0. \quad (10)$$

For such an external force, if (8) is obtained then a natural choice for P_{PART} is

$$P_{\text{PART}} = m_0 \dot{C}. \quad (11)$$

Combining (8), (9) and (11) gives

$$\begin{aligned} m_0 \ddot{C} &= f_{\text{ext}} + f_{\text{self}} \\ &= f_{\text{ext}} - \kappa \left(\frac{2}{3} g(\ddot{C}, \ddot{C}) \dot{C} + \lambda \ddot{C} \right) - \lim_{R_0 \rightarrow 0} \frac{\kappa}{2R_0} \ddot{C}. \end{aligned} \quad (12)$$

Thus we satisfy the orthogonality condition (3) provided $\lambda = -\frac{2}{3}$. By contrast, if (6) is obtained one cannot set

$$m_0 \ddot{C} = f_{\text{ext}} + f_{\text{self}}^{\text{B}}$$

and satisfy (3). Instead an extra term is added ad hoc to the non-electromagnetic contribution to the force in order to compensate for the missing Schott term⁴⁻⁷:

$$\dot{P}_{\text{PART}}^{\text{B}} = m_0 \ddot{C} + \frac{2}{3} \kappa \ddot{C}. \quad (13)$$

With $\lambda = -\frac{2}{3}$ equation (12) gives

$$m_0 \ddot{C} + \lim_{R_0 \rightarrow 0} \frac{\kappa}{2R_0} \ddot{C} = f_{\text{ext}} + \frac{2}{3} \kappa \left(\ddot{C} - g(\ddot{C}, \ddot{C}) \dot{C} \right).$$

The singular coefficient is identified as an electromagnetic contribution to the total mass of the particle, such that the observed mass m satisfies

$$m = m_0 + \lim_{R_0 \rightarrow 0} \frac{\kappa}{2R_0}. \quad (14)$$

The resulting equation of motion for a point charge in an external field is the Lorentz-Abraham-Dirac equation

$$m \ddot{C} = f_{\text{ext}} + \frac{q^2}{6\pi\epsilon_0} (\ddot{C} - g(\ddot{C}, \ddot{C}) \dot{C}). \quad (15)$$

II. CALCULATION OF THE SELF FORCE

In the following calculation expression (8) for the self force is obtained by direct integration of the electromagnetic stress-energy-momentum tensor over the side Σ_T of the Bhabha Tube, and hence the ad hoc term in (13) is avoided in the derivation of the Lorentz-Abraham-Dirac equation.

We use a global Lorentzian frame (x^0, x^1, x^2, x^3) on Minkowski spacetime \mathcal{M} with metric

$$g = -dx^0 \otimes dx^0 + dx^1 \otimes dx^1 + dx^2 \otimes dx^2 + dx^3 \otimes dx^3.$$

In the following implicit summation is over $j, k, l = 0, \dots, 3$.

All fields will be regarded as sections of tensor bundles over appropriate domains of \mathcal{M} . Sections of the tangent bundle over \mathcal{M} will be denoted $\Gamma T\mathcal{M}$ while sections of the bundle of exterior p -forms will be denoted $\Gamma \Lambda^p \mathcal{M}$. Sections over spacetime excluding the worldline are written $\Gamma T(\mathcal{M} \setminus C)$ and $\Gamma \Lambda^p(\mathcal{M} \setminus C)$. For any vector field V denote by \tilde{V} the associated 1-form defined by $\tilde{V} = g(V, -)$. The operator d will denote the exterior derivative and i_V the contraction operator with respect to V . The operator \star is the Hodge map associated with metric g .

The Bhabha tube, defined by the parameters $\tau_1, \tau_2 \in I$ and $R_0 > 0$, is given by $\Sigma_1 \cup \Sigma_2 \cup \Sigma_T$ where

$$\begin{aligned} \Sigma_\mu &= \left\{ C(\tau_\mu) + X \mid g(X, X) = 0, \quad -g(X, \dot{C}) < R_0 \right\}, \\ \Sigma_T &= \left\{ C(\tau) + X \mid g(X, X) = 0, \quad -g(X, \dot{C}) = R_0, \quad \tau_1 \leq \tau \leq \tau_2 \right\} \end{aligned} \quad (16)$$

for $\mu = 1, 2$. Here we have used the affine structure of \mathcal{M} to add a vector to a point to give another point in \mathcal{M} . For comparison the Dirac tube is given by $\Sigma_1^D \cup \Sigma_2^D \cup \Sigma_T^D$ where

$$\begin{aligned} \Sigma_\mu^D &= \left\{ C(\tau_\mu) + Y \mid g(Y, \dot{C}) = 0, \quad g(Y, Y) < (R_0^D)^2 \right\}, \\ \Sigma_T^D &= \left\{ C(\tau) + Y \mid g(Y, \dot{C}) = 0, \quad g(Y, Y) = (R_0^D)^2, \quad \tau_1 \leq \tau \leq \tau_2 \right\}. \end{aligned}$$

For every field point x there is a unique point $\tau_r(x)$ at which the worldline intersects the retarded light-cone at x (see FIG. 2),

$$\tau_r : \mathcal{M} \setminus C \rightarrow \mathbb{R}, \quad x \mapsto \tau_r(x). \quad (17)$$

The null vector $X \in \Gamma T(\mathcal{M} \setminus C)$ is given by the difference between the field point $x = (x^0, x^1, x^2, x^3)$ and the worldline point $C(\tau_r(x))$

$$X|_x = x - C(\tau_r(x)), \quad g(X, X) = 0. \quad (18)$$

The vector fields $V, A, \dot{A} \in \Gamma T(\mathcal{M} \setminus C)$ are defined as

$$V|_x = \dot{C}^j(\tau_r(x)) \frac{\partial}{\partial x^j}, \quad A|_x = \ddot{C}^j(\tau_r(x)) \frac{\partial}{\partial x^j} \quad \text{and} \quad \dot{A}|_x = \dddot{C}^j(\tau_r(x)) \frac{\partial}{\partial x^j}, \quad (19)$$

hence from (3)

$$g(V, V) = -1, \quad g(A, V) = 0, \quad \text{and} \quad g(\dot{A}, V) = -g(A, A). \quad (20)$$

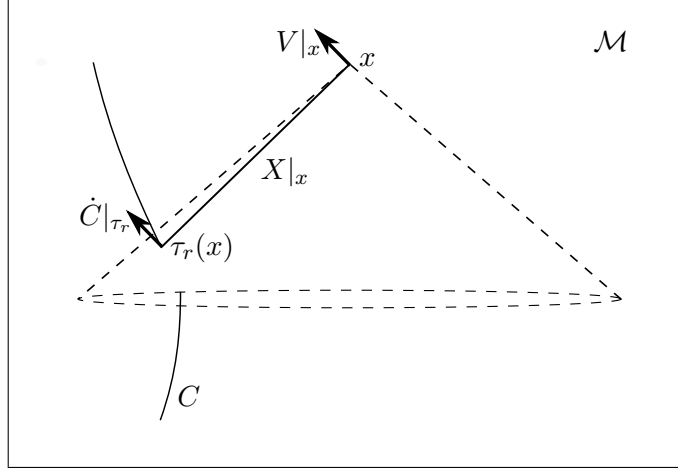


FIG. 2. Retarded time $\tau_r(x)$ and spacetime vectors $V|_x$ and $X|_x$.

The retarded Liénard-Wiechert one-form potential⁹ $\Psi \in \Gamma\Lambda^1(\mathcal{M}\setminus C)$ is given by

$$\Psi = \frac{q}{4\pi\epsilon_0} \frac{\tilde{V}}{g(X, V)}. \quad (21)$$

The corresponding Liénard-Wiechert electromagnetic field $F \in \Gamma\Lambda^2(\mathcal{M}\setminus C)$ is given by

$$F = d\Psi = \frac{q}{4\pi\epsilon_0} \frac{g(X, V)\tilde{X} \wedge \tilde{A} - g(X, A)\tilde{X} \wedge \tilde{V} - \tilde{X} \wedge \tilde{V}}{g(X, V)^3}. \quad (22)$$

It is sometimes beneficial in calculating the self force of a point charge to split F into a radiative and a bound contribution^{1-3,6,10}. There are two common splittings which can be made that are independently motivated, however as emphasized by Gal'tsov and Spirin³ they give different radiative and bound contributions to the self force, differing by the Schott term. In this article we avoid splitting F because it is unnecessary in the calculation.

The four electromagnetic stress 3-forms¹¹ $S_k \in \Gamma\Lambda^3(\mathcal{M}\setminus C)$ are given by

$$S_k = \frac{\epsilon_0}{2} (i_{\partial/\partial x^k} F \wedge \star F - i_{\partial/\partial x^k} \star F \wedge F), \quad \widetilde{\star S_k} = \text{T}\left(-, \frac{\partial}{\partial x^k}\right) \quad (23)$$

where T is the symmetric stress-energy-momentum tensor defined on $\mathcal{M}\setminus C$.

It follows from the vacuum Maxwell equations that S_k is closed in $\mathcal{M} \setminus C$, i.e. $dS_k = 0$. Therefore in an arbitrary region $N \subset \mathcal{M} \setminus C$ of spacetime off the worldline

$$\int_{\partial N} S_k = \int_N dS_k = 0. \quad (24)$$

The component of 4-momentum flux $P_k^{(\Sigma)} \in \mathbb{R}$ through an arbitrary three dimensional hypersurface $\Sigma \subset \mathcal{M}$ is defined by

$$P_k^{(\Sigma)} = \int_{\Sigma} S_k. \quad (25)$$

The lightlike hypersurfaces Σ_1 , Σ_2 , and the timelike hypersurface Σ_T (FIG. 1) are defined by (16) and let Σ_1 be negatively oriented with respect to Σ_T and Σ_2 .

The instantaneous 4-momentum derivative at proper time $\tau_0 \in I$ is defined by

$$\dot{P}_k(\tau_0) = \lim_{\substack{R_0 \rightarrow 0 \\ \tau_1 \rightarrow \tau_0 \\ \tau_2 \rightarrow \tau_0}} \left(\frac{1}{\tau_2 - \tau_1} P_k^{(\Sigma_T)} \right) \quad (26)$$

where $P_k^{(\Sigma_T)}$ is given by (25). This definition is justified heuristically as follows. Inspired by (24) we wish to write

$$P_k^{(\Sigma_T)} = P_k^{(\Sigma_2)} - P_k^{(\Sigma_1)} \quad (27)$$

Ignoring the fact that $P_k^{(\Sigma_1)}$ and $P_k^{(\Sigma_2)}$ are both infinite, we assert

$$\dot{P}_k(\tau_0) = \lim_{\substack{R_0 \rightarrow 0 \\ \tau_1 \rightarrow \tau_0 \\ \tau_2 \rightarrow \tau_0}} \left(\frac{1}{\tau_2 - \tau_1} \left(P_k^{(\Sigma_2)} - P_k^{(\Sigma_1)} \right) \right) \quad (28)$$

Inserting (27) into (28) yields (26).

We define the vector $\dot{P}_{\text{EM}}(\tau_0) \in T_{C(\tau_0)}\mathcal{M}$ by

$$\dot{P}_{\text{EM}}(\tau_0) = \dot{P}_k(\tau_0) g^{kl} \frac{\partial}{\partial x^l} \quad (29)$$

where $g^{kl} = g^{-1}(dx^k, dx^l)$ and g^{-1} is the inverse metric on \mathcal{M} . Since τ_0 is arbitrary there is an induced vector field \dot{P}_{EM} on the curve C .

To simplify the problem we introduce the retarded ‘Newman-Unti’¹² coordinates (τ, R, θ, ϕ) , where $\tau \in I$, $R > 0$, $0 < \theta < \pi$ and $0 < \phi < 2\pi$. The coordinate transformation between (τ, R, θ, ϕ) and (x^0, x^1, x^2, x^3) is given by:

$$\begin{aligned} x^0 &= C^0(\tau) + \frac{R}{\alpha}, \\ x^1 &= C^1(\tau) + \frac{R}{\alpha} \sin(\theta) \cos(\phi), \\ x^2 &= C^2(\tau) + \frac{R}{\alpha} \sin(\theta) \sin(\phi), \\ x^3 &= C^3(\tau) + \frac{R}{\alpha} \cos(\theta) \end{aligned} \quad (30)$$

where $\alpha \in \Gamma\Lambda^0\mathcal{M}$ is given by

$$\alpha(\tau, \theta, \phi) = \frac{g(X, \dot{C}(\tau))}{g(X, \partial_{x^0})} = \dot{C}^0(\tau) - \dot{C}^1(\tau) \sin(\theta) \cos(\phi) - \dot{C}^2(\tau) \sin(\theta) \sin(\phi) - \dot{C}^3(\tau) \cos(\theta) \quad (31)$$

From (30) and (31) it follows

$$R = -g(X, \dot{C}(\tau)) \quad \text{and} \quad \tau = \tau_r(x(\tau, R, \theta, \phi)). \quad (32)$$

The spherical coordinates θ and ϕ are deduced from the global Lorentzian frame (x^0, x^1, x^2, x^3) .

In this coordinate system the metric is given by

$$g = \left(2R\frac{\dot{\alpha}}{\alpha} - 1\right)d\tau \otimes d\tau - (d\tau \otimes dR + dR \otimes d\tau) + \frac{R^2}{\alpha^2}(d\theta \otimes d\theta + \sin^2(\theta)d\phi \otimes d\phi)$$

and the vectors X , V and A are written

$$X = R\frac{\partial}{\partial R}, \quad (33)$$

$$V = \frac{\partial}{\partial \tau} + \frac{\dot{\alpha}}{\alpha}R\frac{\partial}{\partial R} \quad (34)$$

and

$$A = \frac{\dot{\alpha}}{\alpha}\frac{\partial}{\partial \tau} + \frac{\dot{\alpha}}{\alpha}\left(\frac{R\dot{\alpha}}{\alpha} - 1\right)\frac{\partial}{\partial R} + \frac{\dot{\alpha}\alpha_\phi - \alpha\dot{\alpha}_\phi}{R\sin^2(\theta)}\frac{\partial}{\partial \phi} + \frac{\dot{\alpha}\alpha_\theta - \alpha\dot{\alpha}_\theta}{R}\frac{\partial}{\partial \theta} \quad (35)$$

where

$$\dot{\alpha} = \frac{\partial \alpha}{\partial \tau}, \quad \alpha_\theta = \frac{\partial \alpha}{\partial \theta}, \quad \alpha_\phi = \frac{\partial \alpha}{\partial \phi}$$

In the coordinate system Σ_T is given by

$$\Sigma_T = \left\{(\tau, R, \theta, \phi) \middle| \tau_1 \leq \tau \leq \tau_2, \quad R = R_0, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi\right\} \quad (36)$$

Setting $\tau = \tau_0 + \delta$ we expand S_k in powers of δ . We adapt the global Lorentz frame such that

$$\begin{aligned} x^j(C(\tau_0)) &= 0 \quad \text{for } j = 0, 1, 2, 3 \\ \text{and} \quad \dot{C}(\tau_0) &= \frac{\partial}{\partial x^0}, \quad \ddot{C}(\tau_0) = a\frac{\partial}{\partial x^3}, \quad \ddot{C}(\tau_0) = b^j\frac{\partial}{\partial x^j} \end{aligned} \quad (37)$$

where $a, b^j \in \mathbb{R}$ are constants given by

$$a = \sqrt{g(\ddot{C}(\tau_0), \ddot{C}(\tau_0))}, \quad b^j = dx^j(\ddot{C}(\tau_0)) \quad (38)$$

and from (3)

$$b^0 = a^2$$

Thus expanding \dot{C} and \ddot{C} we have

$$\begin{aligned}\dot{C}(\delta + \tau_0) &= \left(1 + \frac{b^0}{2}\delta^2\right)\frac{\partial}{\partial t} + \frac{b^1}{2}\delta^2\frac{\partial}{\partial x} + \frac{b^2}{2}\delta^2\frac{\partial}{\partial y} + \left(a\delta + \frac{b^3}{2}\delta^2\right)\frac{\partial}{\partial z} + \mathcal{O}(\delta^3), \\ \ddot{C}(\delta + \tau_0) &= b^0\delta\frac{\partial}{\partial t} + b^1\delta\frac{\partial}{\partial x} + b^2\delta\frac{\partial}{\partial y} + \left(a + b^3\delta\right)\frac{\partial}{\partial z} + \mathcal{O}(\delta^2)\end{aligned}\tag{39}$$

From (19) and (32) we have

$$V|_{(\delta+\tau_0, R, \theta, \phi)} = \dot{C}(\delta + \tau_0) \quad \text{and} \quad A|_{(\delta+\tau_0, R, \theta, \phi)} = \ddot{C}(\delta + \tau_0)\tag{40}$$

It is useful to express V and A in mixed coordinates, with the basis vectors in terms of the global Lorentz coordinates, but the coefficients expressed in terms of the Newman-Unti coordinates.

We use the MAPLE script^{13,14} that accompanies this article to evaluate the integral of S_k over the side Σ_T of the Bhabha tube. We substitute (28) and (29) into (16) in order to obtain an explicit expression for the Liénard-Wiechert potential Ψ in Newman-Unti coordinates (30). Taking the exterior derivative we obtain the field 2-form F and its Hodge dual $\star F$. We obtain expressions for the four translational Killing vectors $\frac{\partial}{\partial x^k}$ in Newman-Unti coordinates and using (23) we obtain expressions for the four electromagnetic stress 3-forms S_k . Substituting the expansions (39) into these expressions we obtain the integrands, and finally using (36) we integrate over Σ_T . The result is

$$\begin{aligned}\frac{1}{\kappa} \int_{\Sigma_T} S_0 &= -\frac{1}{4}b^0\frac{\delta_2^2 - \delta_1^2}{R_0} - \frac{2}{3}a^2(\delta_2 - \delta_1) - \frac{2}{3}ab^3(\delta_2^2 - \delta_1^2) + \mathcal{O}(\delta_1^3) + \mathcal{O}(\delta_2^3), \\ \frac{1}{\kappa} \int_{\Sigma_T} S_1 &= \frac{1}{4}b^1\frac{\delta_2^2 - \delta_1^2}{R_0} + \mathcal{O}(\delta_1^3) + \mathcal{O}(\delta_2^3), \\ \frac{1}{\kappa} \int_{\Sigma_T} S_2 &= \frac{1}{4}b^2\frac{\delta_2^2 - \delta_1^2}{R_0} + \mathcal{O}(\delta_1^3) + \mathcal{O}(\delta_2^3), \\ \frac{1}{\kappa} \int_{\Sigma_T} S_3 &= \frac{1}{4}b^3\frac{\delta_2^2 - \delta_1^2}{R_0} + \frac{1}{2}a\frac{\delta_2 - \delta_1}{R_0} + \frac{1}{3}a^3(\delta_2^2 - \delta_1^2) + \mathcal{O}(\delta_1^3) + \mathcal{O}(\delta_2^3)\end{aligned}\tag{41}$$

where

$$\delta_1 = \tau_1 - \tau_0, \quad \delta_2 = \tau_2 - \tau_0$$

and κ is given by (5).

Combining (41) into a single expression and using (26) and (29) we obtain the following expression for $\dot{P}(\tau_0) \in T_{C(\tau_0)}\mathcal{M}$

$$\frac{1}{\kappa}\dot{P}(\tau_0) = \frac{2}{3}a^2\frac{\partial}{\partial x^0} + \lim_{R_0 \rightarrow 0} \frac{1}{2R_0}a\frac{\partial}{\partial x^3} + \lim_{\substack{R_0 \rightarrow 0 \\ \tau_1 \rightarrow \tau_0 \\ \tau_2 \rightarrow \tau_0}} \left(\frac{\tau_1 + \tau_2 - 2\tau_0}{4R_0} \right) b^j \frac{\partial}{\partial x^j} + \mathcal{O}(\delta_1^2) + \mathcal{O}(\delta_2^2).$$

Hence from (7) and (37)

$$\frac{1}{\kappa}\dot{P}(\tau_0) = \frac{2}{3}g(\ddot{C}(\tau_0), \ddot{C}(\tau_0))\dot{C}(\tau_0) + \lim_{R_0 \rightarrow 0} \frac{1}{2R_0}\ddot{C}(\tau_0) + \lambda\ddot{C}(\tau_0) + \mathcal{O}(\delta_1^2) + \mathcal{O}(\delta_2^2). \quad (42)$$

The first term in (42) is the standard radiation reaction term and the second term is the singular term to be renormalized. The third term is proportional to $\ddot{C}(\tau_0)$ and therefore may be recognised as the Schott term providing the coefficient is well defined in the limit.

If λ is chosen to be finite it follows immediately that all higher order terms in the series vanish. This is because R_0^{-1} is the most divergent power of R_0 appearing in the series. Mathematically we are free to choose λ to diverge, in which case higher order terms could be made finite. However this would require extra renormalization in order to accommodate the λ terms and the resulting equation of motion would not resemble the Lorentz-Abraham-Dirac equation.

Choosing λ to be finite yields for $\dot{P}_{\text{EM}}(\tau) \in T_{C(\tau)}\mathcal{M}$

$$\frac{1}{\kappa}\dot{P}_{\text{EM}} = \frac{2}{3}g(\ddot{C}, \ddot{C})\dot{C} + \lambda\ddot{C} + \lim_{R_0 \rightarrow 0} \frac{1}{2R_0}\ddot{C}. \quad (43)$$

The value of λ may be fixed by satisfying the orthogonality condition (3),

$$0 = \frac{1}{\kappa}g(\dot{P}_{\text{EM}}, \dot{C}) = -\frac{2}{3}g(\ddot{C}, \ddot{C}) + \lambda g(\ddot{C}, \dot{C}) = -(\frac{2}{3} + \lambda)g(\ddot{C}, \ddot{C}). \quad (44)$$

Therefore $\lambda = -\frac{2}{3}$ and the final covariant expression for f_{self} is given by

$$f_{\text{self}} = -\dot{P}_{\text{EM}} = \frac{2}{3}\kappa(\ddot{C} - g(\ddot{C}, \ddot{C})\dot{C}) - \lim_{R_0 \rightarrow 0} \frac{\kappa}{2R_0}\ddot{C}, \quad (45)$$

which is identical to (4). Thus the complete self force has been obtained without the addition of an ad hoc term to the non-electromagnetic momentum of the particle.

CONCLUSION

It has been shown the complete self force may be obtained directly from the electromagnetic stress-energy-momentum tensor when using the Bhabha tube as the domain of

integration. This eliminates the need to introduce the extra ad hoc term in (13). It also proves the reason for the missing term in previous calculations is the procedure followed in taking the limits, and not the nature of the coordinates used as proposed by Gal'tsov and Spirin³. We have seen that a requirement for the term to appear is that the ratio of limits λ , which describes the way in which the Bhabha tube is collapsed onto the worldline, is made finite. This is a natural choice because it demands δ_1 , δ_2 and R_0 to be the same order of magnitude. The specific value $\lambda = -\frac{2}{3}$ is fixed by the orthogonality condition (3), however the physical justification for imposing this particular geometry on the Bhabha tube is currently unknown.

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